1

The dynamic part of the pressure eqn is

$$\nabla^2 p_D = -\frac{|\underline{def}|^2}{2} + \frac{|\underline{\omega}|^2}{2} = \frac{1}{2}\left(|\underline{\omega}|^2 - |\underline{def}|^2\right)$$

where

$$|\underline{def}|^2 = 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial w}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2$$
$$+ \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2$$

and

$$|\underline{\omega}|^2 = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)^2$$

---

a

Here our wind vector $\underline{u} = (u, v, w)$ is given by the background shear; i.e.,

$$\underline{u} = \underline{U}(z) = \Lambda z \hat{x} = (\Lambda z, 0, 0)$$

The only nonzero terms in $|\underline{def}|^2$ and $|\underline{\omega}|^2$ are thus the $\partial u/\partial z$ terms; that is

$$|\underline{def}|^2 = \left(\frac{\partial u}{\partial z}\right)^2 = |\underline{\omega}|^2$$

s.t.

$$\nabla^2 p_D = \frac{1}{2}\left(|\underline{\omega}|^2 - |\underline{def}|^2\right) = 0$$

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b

In this case we just have the updraft

$$\underline{u}(z) = w(x,y)\hat{z} = \left(0, 0, w_0 e^{-x^2-y^2}\right)$$

so we just have to worry about the $\partial w/\partial x$ and $\partial w/\partial y$ terms; i.e.,

$$|\underline{def}|^2 = \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = |\underline{\omega}|^2$$

So again we have

$$\nabla^2 p_D = 0$$

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c

Putting the shear and updraft together, we have

$$\underline{u} = \Lambda z \hat{x} + w_0 e^{-x^2-y^2}\hat{z} = \left(\Lambda z, 0, w_0 e^{-x^2-y^2}\right)$$

Subbing in then tells us

$$|def|^2 = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial z}\right)^2 + 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2$$

$$|\underline{\omega}|^2 = \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)^2$$

$$= \left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2$$

s.t.

$$|\underline{\omega}|^2 - |def|^2 = -4\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} = -4\Lambda\left(-2x w_0 e^{-x^2-y^2}\right)$$

$$= 8\Lambda x w_0 e^{-x^2-y^2}$$

On the east side of the updraft $(x > 0)$ we have $|\underline{\omega}|^2 > |def|^2$, while on the west side $(x < 0)$ we have $|def|^2 > |\underline{\omega}|^2$

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d

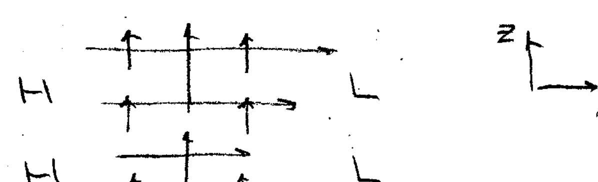
Our rule of thumb is that the curvature $\nabla^2 p_D$ tends to be opposite in sign to the local anomaly in p_D. From the above we know

$$\nabla^2 p_D \begin{cases} > 0 & \text{for } x > 0 \\ < 0 & \text{for } x < 0 \end{cases}$$

which suggests

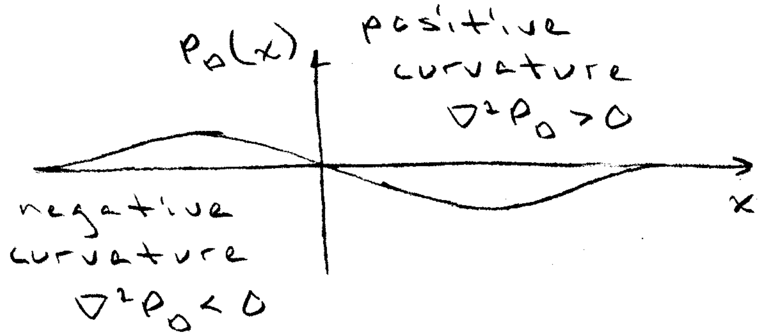
$$p_D \begin{cases} \text{lower for } x > 0 & \text{(east side)} \\ \text{higher for } x < 0 & \text{(west side)} \end{cases}$$

Specifically, we might expect $p_D(x)$ to look



d [cont]

something like



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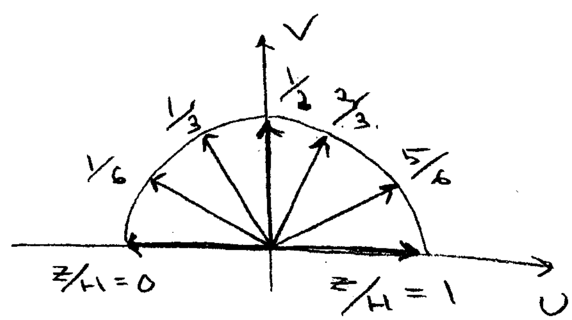
e

For our more realistic background shear profile we use

$$\underline{U}(z) = \begin{cases} -U_0 \cos\left(\frac{\pi z}{H}\right)\hat{x} + U_0 \sin\left(\frac{\pi z}{H}\right)\hat{y}, & \text{for } z \leq H \\ U_0 \hat{x}, & \text{for } z > H \end{cases}$$

The hodograph for this profile is as shown, with the wind turning clockwise with height.



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f

With the updraft included, our net wind vector is

$$\underline{u} = \left(-U_0 \cos\left(\frac{\pi z}{H}\right), U_0 \sin\left(\frac{\pi z}{H}\right), w_0 e^{-x^2-y^2}\right)$$

Keeping only the nonzero terms, we then have

$$|\text{def}|^2 = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2$$

$$= \left(\frac{\partial u}{\partial z}\right)^2 + 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2$$

$$+ \left(\frac{\partial v}{\partial z}\right)^2 + 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \left(\frac{\partial w}{\partial y}\right)^2$$

$$|\underline{\omega}|^2 = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)^2 + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)^2$$

$$= \left(\frac{\partial w}{\partial y}\right)^2 - 2\frac{\partial v}{\partial z}\frac{\partial w}{\partial y} + \left(\frac{\partial v}{\partial z}\right)^2$$

$$+ \left(\frac{\partial u}{\partial z}\right)^2 - 2\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial x}\right)^2$$

F [cont]

s.t.

$$|\varpi|^2 - |def|^2 = -4\frac{\partial u}{\partial z}\frac{\partial w}{\partial x} - 4\frac{\partial v}{\partial z}\frac{\partial w}{\partial y}$$

$$= -4U_0\frac{\pi}{H}\sin\left(\frac{\pi z}{H}\right)\left(-2xw_0 e^{-x^2-y^2}\right)$$

$$-4U_0\frac{\pi}{H}\cos\left(\frac{\pi z}{H}\right)\left(-2yw_0 e^{-x^2-y^2}\right)$$

$$= 8U_0\frac{\pi}{H}w_0\left(x\sin\left(\frac{\pi z}{H}\right) + y\cos\left(\frac{\pi z}{H}\right)\right)e^{-x^2-y^2}$$

At $z=0$ this gives

$$|\varpi|^2 - |def|^2 = 8U_0\frac{\pi}{H}w_0\, y\, e^{-x^2-y^2}$$

showing that for $y>0$ we have $|\varpi|^2 > |def|^2$ while for $y<0$ we have $|def|^2 > |\varpi|^2$. From our pressure eqn we then have

$$\nabla^2 p_0 \begin{cases} >0 & \text{for } y>0 \\ <0 & \text{for } y<0 \end{cases}$$

suggesting

$$p_0 \begin{cases} \text{lower for } y>0 & (\text{north side}) \\ \text{higher for } y<0 & (\text{south side}) \end{cases}$$

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g

For $z/H = 1$ we have

$$|\varpi|^2 - |def|^2 = -8U_0\frac{\pi}{H}w_0\, y\, e^{-x^2-y^2}$$

showing $|def|^2 > |\varpi|^2$ for $y>0$ and $|\varpi|^2 > |def|^2$ for $y<0$. Then

$$\nabla^2 p_0 \begin{cases} <0 & \text{for } y>0 \\ >0 & \text{for } y<0 \end{cases}$$

suggesting

$$p_0 \begin{cases} \text{higher for } y>0 & (\text{north side}) \\ \text{lower for } y<0 & (\text{south side}) \end{cases}$$

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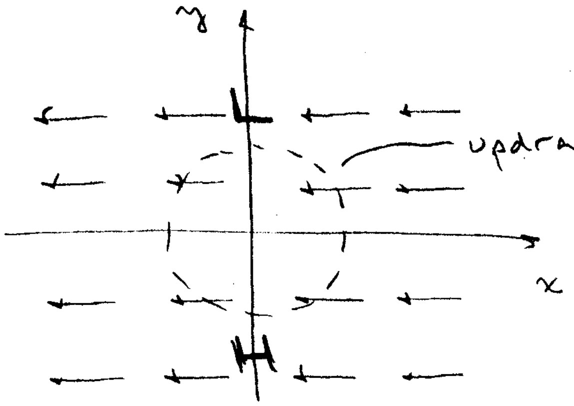
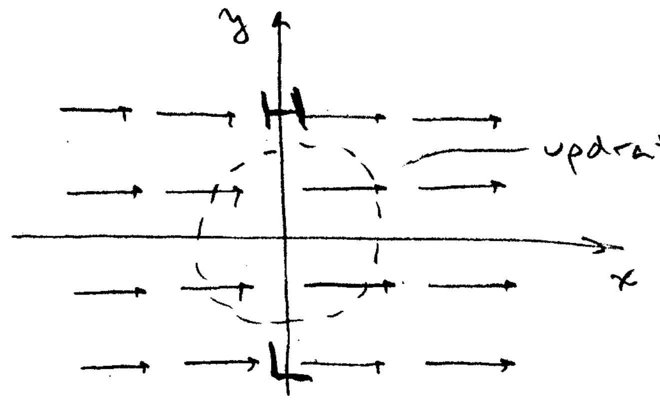
h

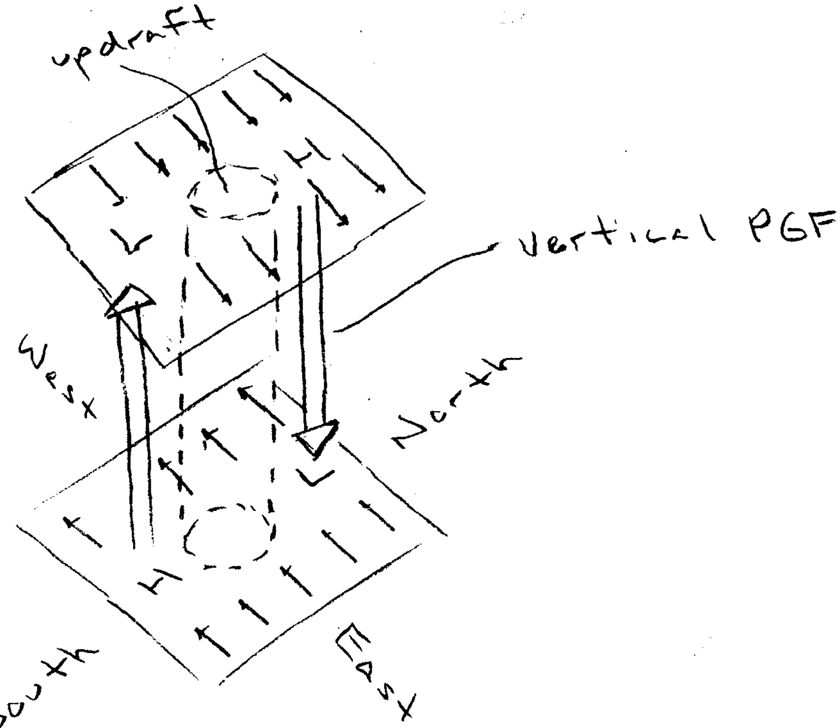


z = 0



z = H

In 3D this might look like



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i

As suggested above, the vertical PGF is upward on the south side and downward on the north side.

j

An upward PGF tends to reinforce the updraft, while a downward force acts against the updraft. So the south side is favored.
