Atmospheric Sciences 435, Spring 2008
Problem Set 5
Due Friday, Apr 4

Problem 1: *The dynamic pressure forcing*
In our diagnostic pressure equation

$$\nabla^2 P = -\frac{|def|^2}{2} + \frac{|\boldsymbol{\omega}|^2}{2} + \frac{\partial b}{\partial z} + f\zeta \tag{1}$$

the first two right-side terms are often called the *dynamic pressure forcing* while the third term is called the *buoyancy forcing*. (The last term is the *planetary forcing* but is typically only important at large scales.) Here the two dynamic forcing terms are explored in greater detail.

I. *The deformation term*
In 336 we showed that a flow with pure stretching deformation looks like

$$u = \frac{T}{2}x \quad , \quad v = -\frac{T}{2}y \tag{2}$$

where $T = \partial u/\partial x - \partial v/\partial y$ is the stretching deformation parameter. We'll ignore the Coriolis terms so that the Boussinesq momentum equations look like

$$\frac{Du}{Dt} = -\frac{\partial P}{\partial x} \qquad \text{and} \qquad \frac{Dv}{Dt} = -\frac{\partial P}{\partial y}$$

where P is the Boussinesq disturbance pressure.

(a) Sketch the flow field defined in (2) over the domain $-1 \le x \le 1$ and $-1 \le y \le 1$.

(b) Suppose that the pressure at $(x,\ y) = (0,\ 0)$ is given by P_0. Assuming the flow field is steady as described by (2), solve for the pressure field as a function of (x, y). (*Hint*: See the second exam solution from 336.....and remember that $Dx/Dt = u$ and $Dy/Dt = v$.)

(c) Is the pressure at the origin a maximum or minimum? Add an H or L to your sketch to indicate the max or min.

(d) Is the sense of the pressure consistent with that expected from (1)?

II. *The vorticity term*
In 2D a pure rotational flow looks like

$$u = -\frac{\zeta}{2}y \quad , \quad v = \frac{\zeta}{2}x \tag{3}$$

where $\zeta = \partial v/\partial x - \partial u/\partial y$ is the vertical vorticity. Again ignore Coriolis and assume that the flow field is steady as described by (3).

(e) Sketch the flow field on the domain $-1 \le x \le 1$ and $-1 \le y \le 1$.

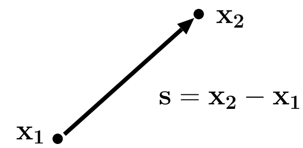
(f) If the pressure at the origin is P_0, solve for the pressure field as a function of $(x,\ y)$.

(g) Is the pressure at the origin a maximum or minimum? Add an H or L to your sketch to indicate the max or min.

(h) Is the sense of the pressure consistent with that expected from (1)?

Problem 2 *The stretching and tilting operator*

Consider two nearby points $\mathbf{x_1}$ and $\mathbf{x_2}$ and let the vector connecting these two points be given by



$$\mathbf{s} = \mathbf{x_2} - \mathbf{x_1} = (\Delta x,\, \Delta y,\, \Delta z)$$

where

$$\Delta x = x_2 - x_1\ , \qquad \Delta y = y_2 - y_1\ , \qquad \text{and} \qquad \Delta z = z_2 - z_1$$

Suppose that $\mathbf{x_1}$ and $\mathbf{x_2}$ are close enough that the components Δx, Δy and Δz can all be considered small.

Note that gradients in the wind between the two points will cause $\mathbf{x_1}$ and $\mathbf{x_2}$ to move at different rates and in different directions. This relative motion of $\mathbf{x_1}$ and $\mathbf{x_2}$ then causes the vector $\mathbf{s}$ to be stretched and tilted with time.

(a) Consider just the x-component Δx and show that the rate of change of Δx can be approximated by

$$\frac{D}{Dt}(\Delta x) = \frac{\partial u}{\partial x}\,\Delta x + \frac{\partial u}{\partial y}\,\Delta y + \frac{\partial u}{\partial z}\,\Delta z = \mathbf{s}\cdot\boldsymbol{\nabla} u$$

That is, the rate of change of Δx is determined solely by the gradient in u between the two points.

(b) Find the equivalent rates of change for the y and z components and show that the three components can be combined into the vector equation

$$\frac{D\mathbf{s}}{Dt} = (\mathbf{s}\cdot\boldsymbol{\nabla})\,\mathbf{u} \tag{4}$$

where

$$(\mathbf{s}\cdot\boldsymbol{\nabla})\,\mathbf{u} = \Big((\mathbf{s}\cdot\boldsymbol{\nabla})\,u\Big)\,\hat{\mathbf{x}} + \Big((\mathbf{s}\cdot\boldsymbol{\nabla})\,v\Big)\,\hat{\mathbf{y}} + \Big((\mathbf{s}\cdot\boldsymbol{\nabla})\,w\Big)\,\hat{\mathbf{z}}$$

The expression (4) describes the stretching and tilting of the vector $\mathbf{s}$ due to the gradient in $\mathbf{u}$ between the points $\mathbf{x_1}$ and $\mathbf{x_2}$. The righthand side of this expression is thus called the *stretching and tilting operator*. Note that this stretching and tilting operator also shows up in the vector vorticity equation, but applied to ω rather than $\mathbf{s}$.

(c) Consider a steady flow field specified by

$$u = 0\ , \qquad v = 0\ , \qquad \text{and} \qquad w = A + x$$

where A is a constant. Suppose that at time $t = 0$ the vector connecting the two nearby points can be written as

$$\mathbf{s} = (\alpha,\, 0,\, 0)$$

where α can be considered small. Solve for $\mathbf{s}$ as a function of time and sketch the result at times $t = 0$, $t = 0.5$, $t = 1$ and $t = 1.5$.