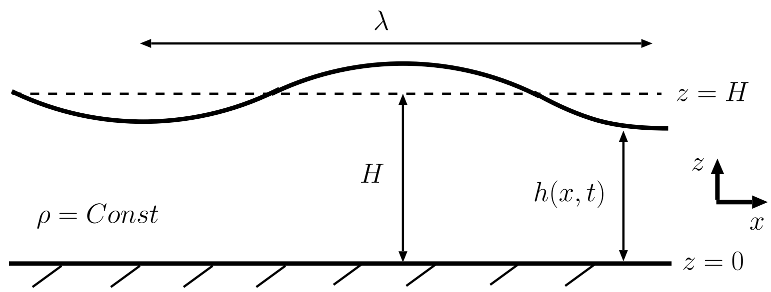
Atmospheric Sciences 435, Spring 2008
Problem Set 2
Due Friday, Feb 15

Problem 1 *The long-wave limit for gravity waves*
Consider a surface wave on a fluid layer of constant density ρ and resting depth H as illustrated below. Let the wavelength of the wave be given by λ and let the depth of the fluid in the disturbed state be $h(x,t)$. For simplicity we assume that the evolution of the wave is fast enough that the Coriolis force can be ignored.



It turns out that the pressure distribution for this type of wave is given by

$$p(x,z,t) = p_a + \rho g(H - z) + \rho g(h - H)\,\frac{e^{kz} + e^{-kz}}{e^{kH} + e^{-kH}}$$

where p_a is the atmospheric pressure and where $k = 2\pi/\lambda$ is the *wavenumber* of the wave. Show that if the wavelength λ of the wave is much greater than the fluid depth H, then the pressure distribution is approximated by

$$p = p_a + \rho g(h - z)$$

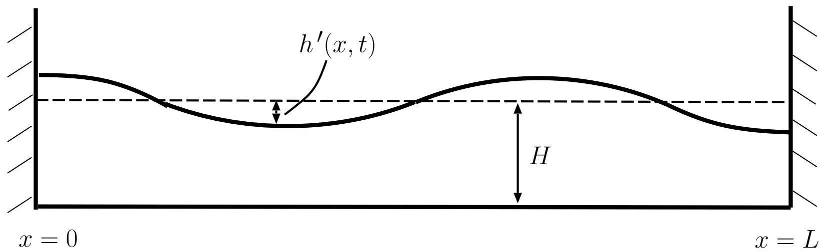
That is, in the long-wave limit the pressure in the wave is approximately the hydrostatic pressure. (*Hint*: What is the Taylor series approximation for e^x when x is small?)

Problem 2 *Long waves on Lake Sommerville*
In class we found that the non-rotating shallow-water system has wave solutions of the form

$$h'(x,t) = \hat{h}\cos(kx - \omega t)$$

where $\hat{h}$ is a constant amplitude. This type of wave is called a *traveling wave*, since the disturbance simply propagates to the right (for positive k and ω) without change of shape. Traveling waves provide a good description for waves in *free space*—that is, away from the confining effects of walls and other boundaries. But in many practical problems the waves are actually constrained (by walls or shores) and our traveling-wave solutions are thus no longer directly applicable.

As a model problem, consider a shallow layer of fluid of resting depth H confined by vertical walls at $x = 0$ and $x = L$, as illustrated below. (Think a lake, lagoon, or even cold air trapped in a valley.) To keep things simple, we'll assume that the Coriolis force can be neglected, that the background state is at rest, and that the disturbance is independent of y.



(a) As done in class, show that for the above set of assumptions the evolution of the disturbance is described by the wave equation

$$\frac{\partial^2 h'}{\partial t^2} = gH \frac{\partial^2 h'}{\partial x^2} \tag{1}$$

where $h'(x,t)$ is the disturbance height.

(b) The difference between the problem here and that considered in class is that in the present case we have to deal with the walls. Specifically, since the fluid cannot move through the walls the disturbance must at all times satisfy the constraint $u' = 0$ at both $x = 0$ and $x = L$. Show that the corresponding condition on h' is that

$$\left.\frac{\partial h'}{\partial x}\right|_{x=0} = \left.\frac{\partial h'}{\partial x}\right|_{x=L} = 0 \tag{2}$$

(c) It should be relatively apparent that a traveling wave cannot at all times satisfy the boundary conditions given in (b). So in place of a traveling wave we'll instead look for so-called *standing wave* solutions that look like

$$h'(x,t) = \hat{h}\cos(kx)\cos(\omega t)$$

where $\hat{h}$ is again a constant amplitude. As with the traveling wave, the standing wave can be made to satisfy the wave equation (1) as long as k and ω satisfy a certain relation. What is this relation?

(d) Show that the solution given in (c) will also satisfy the boundary conditions (2) as long as k is one of a certain set of discrete values. What are these special allowed values of k?

(e) Find the disturbance velocity u' corresponding to your standing wave solution derived in (c) and (d).

(f) Consider the case in which $k = \pi/L$. (This should be one of your allowed values of k derived in (d).) Sketch the disturbance height and velocity for this case on the interval $0 \le x \le L$ at the times $\omega t = 0$, $\pi/2$, π and $3\pi/2$. (For the velocity simply draw arrows at the points of maximum positive and negative u'.)

(g) Consider a lake with depth $H = 12$ m and length $L = 32$ km (roughly the length and depth of nearby Lake Sommerville). What are the oscillation frequency and period associated with the $k = \pi/L$ mode sketched above?

Problem 3 *Which is faster: a shallow-water ocean wave or a commercial jet?*

In class we showed that if we include rotation then the freqency and wavelength of a shallow water gravity wave are related by

$$\omega^2 = gHk^2 + f^2$$

where f is the constant Coriolis parameter. Suppose that we apply this to the case of a long ($\lambda \gg H$) wave propagating on the open ocean. We'll take the depth of the ocean to be roughly 5 km and let the latitude be 45°N.

(a) What is the distance (in km) from New York to London? (*Hint*: Google does nicely here.)

(b) For waves on the deep ocean we can typically ignore the term involving the Coriolis parameter. To see this, set the wavelength equal to the distance described in (a) and then compare the sizes of the gHk^2 and f^2 terms. Would waves with shorter wavelengths then have larger or smaller gHk^2?

(c) Show that if we ignore the Coriolis term then the phase speed of the wave is independent of its wavelength. What is this phase speed for the parameters given above? How long would it take for the wave to cross the Atlantic?

(d) Which is faster: a long ocean wave or a commercial jet?