Integrating the Hypsometric Equation—The Trapezoidal Rule
(Atmo 336, Fall 2007)

Getting Started: In class we showed that the thickness of the layer of air between any two pressure levels p_a and p_b is given by the hypsometric equation

$$z_a - z_b = \frac{R}{g} \langle T \rangle \ln \left(\frac{p_b}{p_a} \right) \tag{1}$$

where implicitly we've let p_a be the top of the layer (lower pressure) and p_b the bottom. The $\langle T \rangle$ term in (1) is the log-weighted average temperature of the layer defined by

$$\langle T \rangle = \frac{\int_{p_a}^{p_b} T \, d(\ln p)}{\int_{p_a}^{p_b} d(\ln p)} \tag{2}$$

where $d(\ln p) = dp/p$.

Typically we use these formulae in class by defining some simple function $T(p)$ to represent the temperature of the atmosphere as a function of pressure and then integrating this function as in (1) and (2) to get the thickness (or distance between the pressure levels) of the layer in question. But in the real world we never actually know the function $T(p)$. What we know instead is a series of discrete observations giving T at a finite set of pressure values between p_a and p_b. To get the thickness of the layer we then need to approximate the hypsometric equation using only this finite set of observations.

A Simple Test Case: We'll start by defining a simple test problem for which the integral defining $\langle T \rangle$ can be evaluated by hand. (This is still dynamics class after all!) We'll use our answer to this problem later when we test our methods for integrating the hypsometric equation using only a finite set of values.

(a) First let's put the hypsometric equation into a form that's more convenient for computation. Verify that substituting (2) into (1) reduces the equation to

$$z_a - z_b = \frac{R}{g} \int_{p_a}^{p_b} \frac{T(p)}{p} \, dp \tag{3}$$

where again we've let p_a be the top of the layer (lower pressure) and p_b the bottom.

(b) Now to define the temperature profile. Let's suppose

$$T(p) = T_s - \alpha \left(1 - \frac{p}{p_s} \right) + \beta \cos \left(\frac{p}{p_s} \frac{\pi}{2} \right) \frac{p}{p_s} \tag{4}$$

where T_s is the surface temperature, p_s is the surface pressure, and α and β are specified constants. This looks a bit complicated, so let's first plot the function to get a basic idea of what it looks like. Start by making a function m-file of the form

```
function y = Tdef(p)
Ts = ;
ps = ;
alph = ;
bet = ;
y = Ts - alph.*(1.-p./ps) + bet.*cos(p.*pi./2./ps).*p./ps;
```

and saving the file as *Tdef.m*. The values of the constants will of course be filled in later. Then to make the plot you would type on the command line (or in a separate m-file)

```
> pa = ;
> pb = ;
> p = pa:2.:pb;
> T = Tdef(p);
> plot(T,p);
> axis ij;
```

where the last command ensures that the pressure decreases along the vertical axis rather than increases.

Plot the temperature profile between $p_a = 500$ hPa and $p_b = 1000$ hPa for the values $T_s = 300$ K, $p_s = 1000$ hPa, and $\alpha = 73$ K. First plot the case $\beta = 0$ for reference and then overlay as a red line (using *plot(T,p,'r');*) the case $\beta = 40$ K. Save your result as *T_profile.jpg* and copy to the figure directory to turn in.

(c) Ok, now for the derivation. Show that substituting our temperature profile (4) into (3) results in the layer thickness being given by

$$z_a - z_b = \frac{R}{g}\left[(T_s - \alpha)\ln\frac{p_b}{p_a} + \frac{\alpha}{p_s}(p_b - p_a) + \frac{2\beta}{\pi}\sin\left(\frac{p}{p_s}\frac{\pi}{2}\right)\Big|_{p_a}^{p_b}\right] \tag{5}$$

where of course the last term is evaluated at p_b and p_a. What does the 1000 to 500 hPa thickness turn out to be for the values given above (with $\beta = 40$ K)? Record your answer to 2 decimal places (or six significant digits). We'll use this as a baseline for testing the approximation methods that follow.

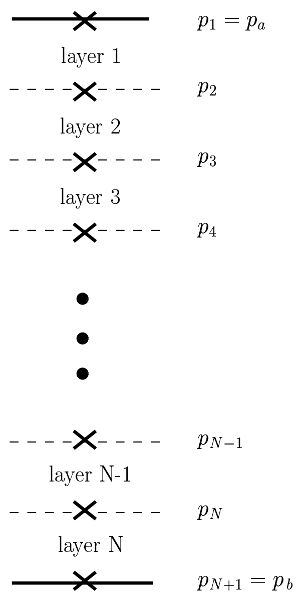
Approximating the Integral: Now suppose that we don't actually know that the temperature profile is described by (4). (It still is—we just don't know that.) Instead all we are given is a set temperature measurements at some finite set of pressure values between p_a and p_b (as returned from a radiosonde or some other intrument, for example). But our boss really wants to know the 1000 to 500 hPa layer depth—and to within 1 m. So what do we do? Well, we approximate the integral of course.

The simplest way to see how to approximate the integral (3) is to recall that an integral is nothing more than the sum over a number of thin slices in the limit that the slice thickness decreases to zero. That is, symbolically

$$\frac{R}{g}\sum_{k=1}^{N}\frac{T(p_k)}{p_k}\,\Delta p \longrightarrow \frac{R}{g}\int_{p_a}^{p_b}\frac{T(p)}{p}\,dp$$

in the limit that the slice thickness Δp becomes very thin—or equivalently in the limit that the number of slices N between p_a and p_b becomes very large. To appoximate the integral we'll just turn this around. That is, the integral is approximated by a sum over N slices in the limit that the number of slices is large.

Now to be more precise: suppose that we have $N+1$ observations of temperature equally spaced in pressure between p_a and p_b. Suppose that the observations include values at both p_a and p_b. These $N+1$ observation points can then be used to define a set of N sublayers between p_a and p_b, as illustrated below.



Having broken the p_a to p_b layer into sublayers, the question then just becomes: how do we approximate T and p for each layer?

Well, the simplest approach is just to average the the values at the top and bottom of the layer. That is, for layer 1 we approximate $p \approx (p_1 + p_2)/2$ and $T \approx (T_1 + T_2)/2$, and so on. Making this approximation for each of our sublayers and then summing gives

$$\frac{R}{g} \int_{p_a}^{p_b} \frac{T(p)}{p} \, dp \approx \frac{R}{g} \sum_{k=1}^{N} \frac{T(p_k) + T(p_{k+1})}{p_k + p_{k+1}} \, \Delta p \tag{6}$$

where $\Delta p = (p_b - p_a)/N$. This scheme for approximating the integral given a set of $N+1$ observations is known as the *trapezoidal rule* for numerical integration.

Computations: Ok, now to try out our method. For each of the following compute the layer depth between 1000 and 500 hPa given the temperature profile (4) with the values of T_s, p_s, α and β as given above (with $\beta = 40$ K).

(a) Compute by hand the trapezoidal approximation for two sublayers. That is, assume that you have observations—returning temperature values as in (4)—at the pressure levels 500, 750 and 1000 hPa. How good is your approximation in this case?

Record your answer in terms of the fractional error

$$E = \frac{D_{true} - D_{approx}}{D_{true}}$$

where D is layer depth and D_{true} is given by (5). Have we satisfied the boss with this calculation?

(b) Now let's try the trapezoidal approximation for $N = 4$. In this case computing the approximation by hand will be a bit too tedious (for me anyway), so we'll instead write a little program to do it for us (which you should save as an m-file):

```
N = 4;
pa = 500;
pb = 1000;
dp = (pb-pa)./N;
p = pa:dp:pb;
T = Tdef(p);

sum = 0.;
for k=1:N
    sum = sum + ;
end

truval = ;
err = (truval-sum)./truval
```

What is the fractional error in this case? Has increasing the layers improved the approximation?

(c) Compute the approximation for $N = 2, 3, 4, 6, 8, 10, 16, 32,$ and 64 and record the fractional error in each case. Then plot the errors as a function of Δp in MATLAB. Save your result as *error_plot.jpg* and copy to the figure directory to turn in. Is the trapezoidal approximation approaching the true integral for $\Delta p \to 0$? (That is, does the fractional error go to zero in this limit?) And have we satisfied the boss?