Mr. Taylor and His Series
(Atmo 336, Fall 2007)

Getting Started: In class we showed that the density of dry air with pressure p and temperature T is given by the ideal gas law

$$\rho_d = \frac{p}{R_d T} \qquad \text{(dry air)}$$

while the density of moist air at the same pressure and temperature is given by

$$\rho = \frac{p}{R_d T} \, \frac{\epsilon/r + \epsilon}{1 + \epsilon/r} = \frac{p}{R_d T} \, \epsilon \, \frac{1 + r}{r + \epsilon} \, . \qquad \text{(moist air)}$$

Here r is the mixing ratio of water vapor and $\epsilon \approx 0.62$ is the ratio of the gas constants R_d / R_v. To get some basic insight into the effect of water vapor on ρ, let's take a look at the ratio

$$\sigma = \frac{\rho}{\rho_d} = \epsilon \, \frac{1 + r}{r + \epsilon}$$

which we refer to as the *moisture correction* for density (i.e., to get the real density ρ we multiply the density for dry air ρ_d by the moisture correction σ).

First let's plot the moisture correction in MATLAB. Note that a mixing ratio of $r = 0.33$ implies that a full quarter of the mass of the air is water vapor. (That is, the mass of water vapor is one-third as large as the mass of dry air, suggesting a 25% / 75% split.) This is much larger than any normally realized value in the atmosphere, so we'll start with $r = 0.33$ as the right edge of our domain in r. We can then plot σ in MATLAB by issuing the following set of commands:

```
> r = 0.:0.01:0.33;
> eps = 0.62;
> sig = eps.*(1+r)./(r+eps);
> plot(r,sig,'r-','linewidth',2);
> xlabel('mixing ratio','fontsize',16);
> ylabel('moisture correction','fontsize',16);
> axis([0 0.33 0.8 1]);
> set(gca,'ytick',0.8:0.025:1);
> set(gca,'xtick',0:0.05:0.35);
```

The first command defines a vector r containing real values evenly spaced (with interval 0.01) between 0 and 0.33. The next three lines then define the function sig and plot it using a red line with a thickness of 2. The remaining lines then add axis labels and set axis properties. Save your plot as a jpg file named *sigma_r.jpg* and copy to your personal figure directory to turn in. Then ponder the following questions (no need to write down answers—just ponder):

- Does adding water vapor tend to increase or decrease the density relative to that of dry air?
- Your plot should show that the moisture correction at $r = 0$ is 1. Does this make sense?
- Where is the change in density with r most rapid: near $r = 0$ or $r = 0.33$?
- How large does r need to be to produce a change in the density of 10% (i.e., a moisture correction of 0.90)?

Introducing Mr. Taylor: Brook Taylor was an English mathematician who expanded on the work of Isaac Newton. His famous formula for approximating an arbitrary function by a polynomial (i.e., a *Taylor series*) first appeared in the *Journal of Early Calculus Results* in 1715. In 1715 MATLAB was still two and a half centuries away and mathematicians and physicists thus had a very strong incentive to simplify problems as much as possible in order to gain insight. Let's follow in the footsteps of Mr. Taylor by approximating the moisture correction $\sigma(r)$ by a Taylor series for small r.

(a) To get started, let's first rewrite σ in the form

$$\sigma(r) = (1+r)\, g(r) \qquad \text{where} \qquad g(r) = \epsilon\,(\epsilon + r)^{-1}$$

(verify that this is the same as the formula given above). Next we approximate $g(r)$ for small values of r by a polynomial

$$g(r) \approx a_0 + a_1\, r + a_2\, r^2 + \dots\ .$$

How are the coefficients a_k related to the value and derivatives of $g(r)$ at $r = 0$? Show that the end result is

$$g(r) \approx 1 - \frac{1}{\epsilon}\, r + \frac{1}{\epsilon^2}\, r^2 + \dots$$

(b) Next we'll multiply the result above by $(1 + r)$ to get σ; i.e.,

$$\begin{aligned} \sigma(r) &= (1+r)\, g(r) \\ &\approx (1+r)\left(1 - \frac{1}{\epsilon}\, r + \frac{1}{\epsilon^2}\, r^2 + \dots\right) \end{aligned}$$

Show that grouping terms of like power then gives a polynomial approximation

$$\sigma(r) \approx b_0 + b_1\, r + b_2\, r^2 + \dots\ . \tag{1}$$

What are the first three coefficients b_0, b_1, and b_2?

(c) Now to plot the fruits of our labor. First let's repeat our prior steps to plot the total moisture correction σ

```
> clear;
> r = 0.:0.01:0.33;
> eps = 0.62;
> sig = eps.*(1+r)./(r+eps);
> plot(r,sig,'r-','linewidth',2);
> xlabel('mixing ratio','fontsize',16);
> ylabel('moisture correction','fontsize',16);
> axis([0 0.33 0.8 1]);
> set(gca,'ytick',0.8:0.025:1);
> set(gca,'xtick',0:0.05:0.35);
> hold on;
```

where the command *hold on* ensures that subsequent plots will be overlayed on the current plot rather than replacing them (you can reverse this with *hold off*). Next let's define our linear polynomial approximation $\sigma(r) \approx b_0 + b_1 r$

```
> b0 =
> b1 =
> taylor_1 = b0 + b1.*r;
> plot(r,taylor_1,'b--','linewidth',2);
```

(where you should fill in the correct expressions for b0 and b1, of course) which plots the first two terms of the Taylor series approximation as a blue dot-dash line. How good is this linear approximation? At what value of r is the difference between the linear and true curves 0.025 (roughly 2.5% error)? Save your figure to the jpg file *taylor_1.jpg* and copy to the figure directory to turn in.

(d) Next let's pull in the quadratic term so that $\sigma \approx b_0 + b_1 r + b_2 r^2$

```
> b2 =
> taylor_2 = b0 + b1.*r + b2.*r.^2;
> plot(r,taylor_2,'g-.','linewidth',3);
```

Again save your plot as a jpg file (this time called *taylor_2.jpg*) and copy to the figure directory. Does including the quadratic term improve things? At what value of r does the error reach 2.5% now?

(e) Typical values of the mixing ratio in the atmosphere are of the order 10^{-3} with a very large value being 0.02. So now let's zoom in on the mixing-ratio range $0 \leq r \leq 0.025$. First clear the figure by typing

```
> clf;
> hold off;
```

Now repeat the steps in (c) and (d) using a re-defined r vector given by 0:0.00025:0.025 and leaving out the *axis* and *set* commands so that the axis limits and tick marks are set automatically. How good are the linear and quadratic Taylor series approximations over this range in r? Save your final plot (with all three of the curves plotted) to the file *taylor_zoom.jpg* and copy to the figure directory.