Weather Observation and Analysis

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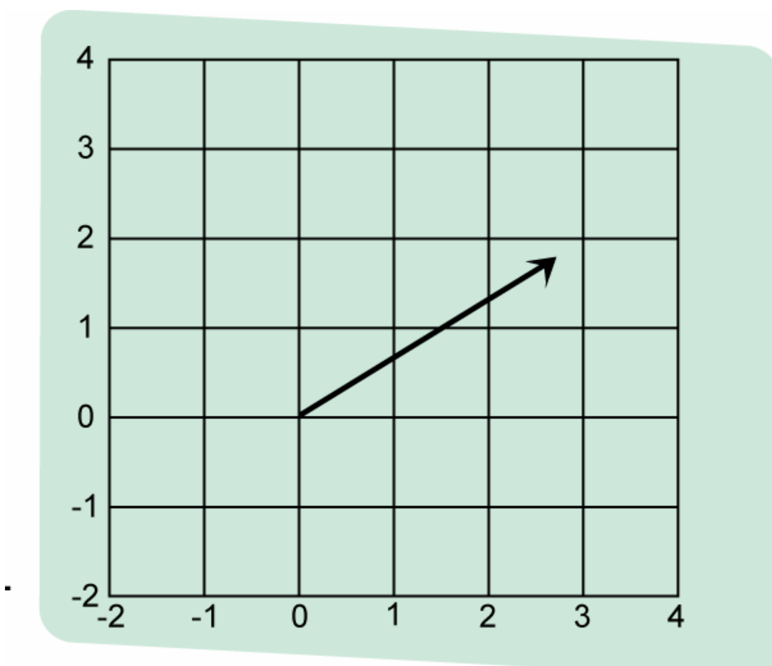
Course Notes

Chapter 7. BITS OF VECTOR CALCULUS

7.1 Vector Magnitude and Direction

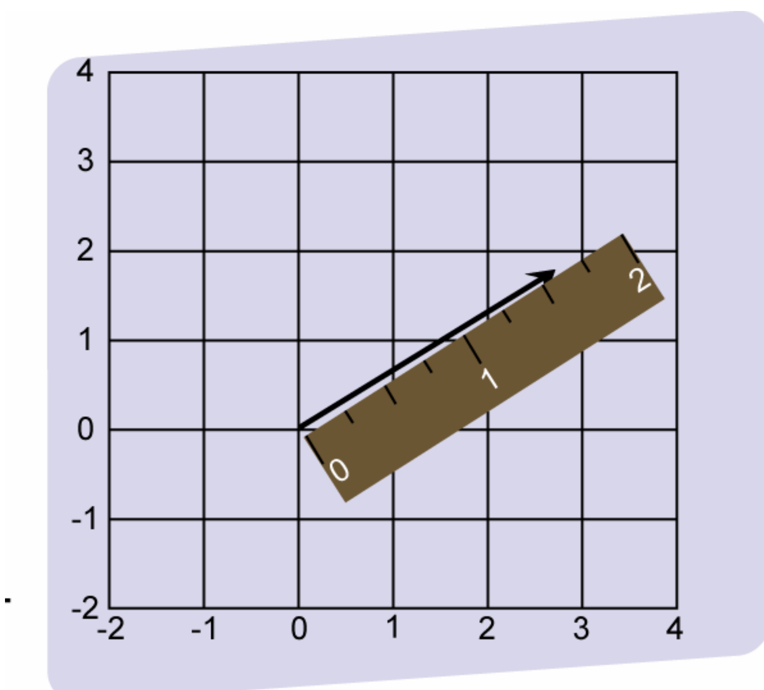
Consider the vector shown in the diagram. The vector is drawn pointing toward the upper right. The origin of the vector is, literally, the origin on this *x*-*y* plot.



Suppose we want to know the magnitude of this vector. In high school you probably learned about computing vector lengths by starting with the magnitudes of the components of the vector, computing the squares of the lengths, summing them, and then taking the square root. Well, we'll use that technique eventually, but that's way too complicated for most weather analysis applications.

Instead, let's keep things simple. The length of the vector is proportional to its magnitude, so once we know what a given vector length corresponds to, we can just measure the vector and convert it to a magnitude. Since the figure in this case has a grid background, we'll start by asking what the magnitude of a vector would be if it were exactly one grid box side long. Then we can see how many grid boxes the vector covers, and convert that to a vector magnitude.

Enough hypotheticals, let's do this for real. Let's say the vector is the horizontal wind. The magnitude of the wind is called the wind speed. Now suppose each grid box corresponds to a wind speed of one meter per second (1 m s^{-1}). If we take a ruler to the page, we find that each grid box is half an inch wide. So a vector that's ½ inch long on this particular graph would have a magnitude of 1 m s^{-1}. A vector that's an inch long would be 2 m s^{-1}, a vector that's 1 ½ inches long would be 3 m s^{-1}, and so forth. If we measure the vector, we find that the vector is 1.6 inches long. So the wind speed is just a little bit more than 3 m s^{-1}. Specifically, it's 3.2 m s^{-1}.



For some of you, it may be obvious where that answer came from. If not, this is like any conversion problem, and solving conversion problems is a necessary skill, so let’s work it out in detail. There is one conversion here:

$$1 \text{ m s}^{-1} = 0.5 \text{ graph inches}$$

In this case, we know the length of the vector in graph inches. Divide the conversion equation by the side with the units that have been measured:

$$1 \text{ m s}^{-1} / 0.5 \text{ graph inches} = 0.5 \text{ graph inches} / 0.5 \text{ graph inches}$$

The right hand side is unity: a number divided by itself. Divide and simplify the left hand side so that its denominator is unity too:

$$2 \text{ m s}^{-1} / \text{graph inch} = 1$$

Now, as you know from arithmetic, you can multiply any number by 1 and get the same number back. Since the left hand side of the above equation is equal to 1, you can multiply your measurement by it to convert units. So, since the drawn vector is 1.6 graph inches, we get

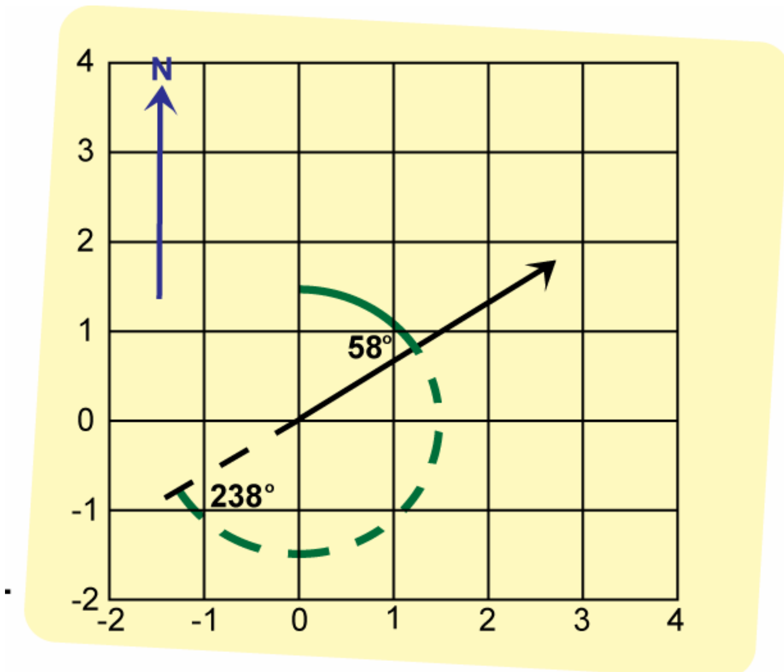
$$\text{vector magnitude} = 1.6 \text{ graph inches} * 2 \text{ m s}^{-1} / \text{graph inch} = 3.2 \text{ m s}^{-1}$$

Okay, that’s reasonably quantitative for the wind speed, so now what about the wind direction? You might think the wind direction is from the southwest, as the vector is drawn, but that depends on which way north is. If you place a piece of paper with the figure on the ground, the vector might actually be pointing south, or northwest, or east-southeast! To give a compass direction, we must first know the orientation of the grid.

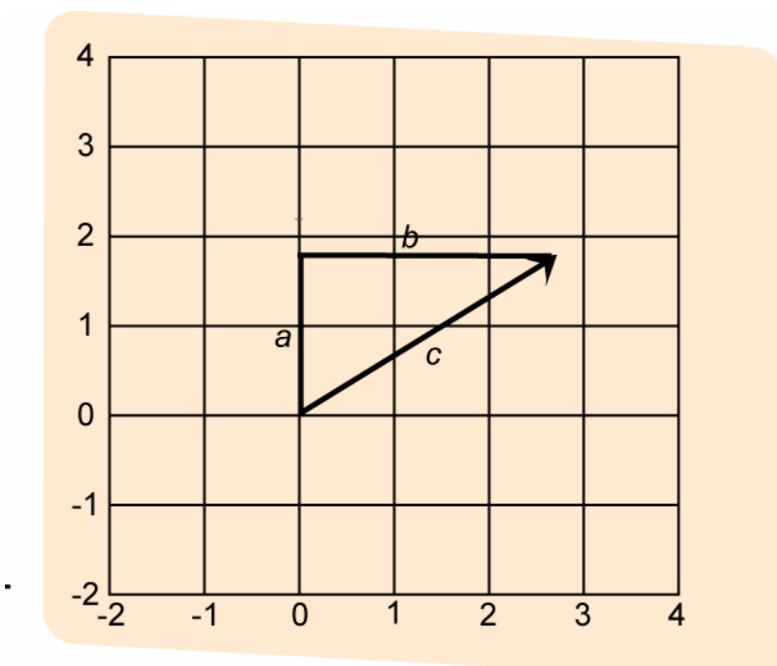
For cartesian coordinates such as these, the conventional orientation has the x axis pointing toward the east and the y axis pointing toward the north. This gives the graph the same orientation as a map, if you normally look at a map with north at the top. With this orientation of the graph specified, the vector is indeed pointing from the southwest.

But hey, “southwest” isn’t good enough. We determined the wind speed to a precision of 0.1 m s^{-1}; surely we owe our estimation of the direction similar care. So if you take out your protractor (that is, if you have a protractor or remember what one is), and measure the angle relative to north (the y axis), you will find that the wind vector is pointing toward a compass heading of about 58 degrees (see Chapter 3 for the correspondence between cardinal directions and degrees).

For most vectors, that might be a good enough answer, but remember (again, from Chapter 3) that wind is an exception. Meteorologists express wind direction as the direction the wind is coming from, not going towards. So we must add 180 degrees to get the compass heading on the opposite side of the compass dial: this wind direction is 238 degrees.



Suppose you don't have a protractor. With a calculator, you can still compute the orientation, with just a little trigonometry. Look at the next figure, which shows the triangle formed by the vector relative to the axis representing north. The tangent of the angle we are looking for is equal to *b/a*. And the values of *b* and *a* are easily seen from the figure: *b*,

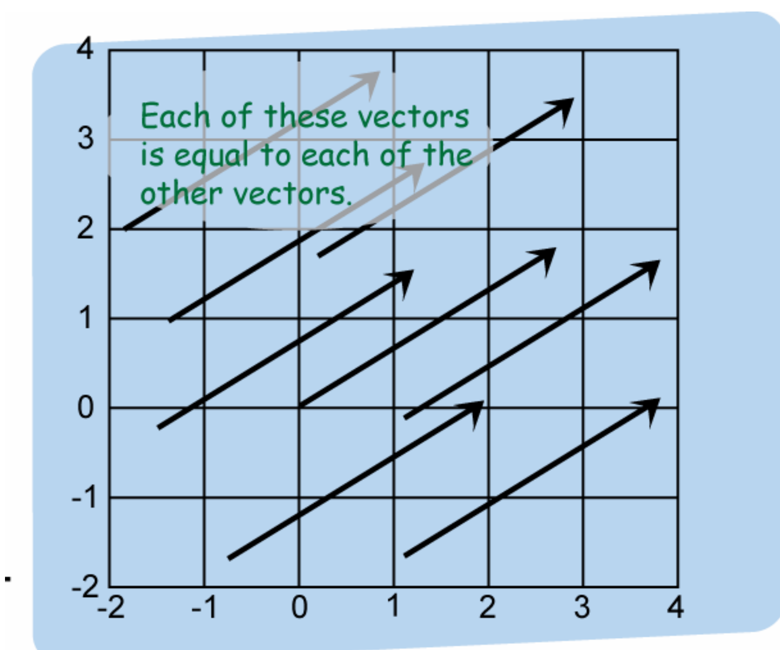


the distance of the endpoint of the vector from the *y* axis, is 2.8 grid boxes (1.4 m s^{-1}), using the conversion we worked out earlier, and *a*, the distance of the endpoint of the vector from the *x* axis, is 1.8 grid boxes (0.9 m s^{-1}). Plugging these values into a calculator, the arctangent of 2.8/1.8 (or 1.56) is 57 degrees, mighty close to the previous estimate. Add 180 to that, and we get a wind direction of 237, which is equal to 238 to within the level of accuracy of our estimate.

Without a ruler, we could also have used simple trigonometry to determine the magnitude of the wind. By the Pythagorean Theorem, $a^2 + b^2 = c^2$, the square of the length of the vector. A little more calculator work tells us that $(0.9 \text{ m s}^{-1})^2 + (1.4 \text{ m s}^{-1})^2 = 2.77 \text{ m}^2 \text{ s}^{-2}$, so *c* is the square root of 2.77 m^2 s^{-2}, or1.68 m s^{-1}. This, too, is pretty close to our ruler estimate.

To summarize so far, we have determined the approximate value of this wind vector to be 1.6 m s^{-1} from 238 degrees, based on its length and orientation. To do that, we needed to know the length scale and orientation of the graph.

The vector we worked with began at the origin of the graph. But neither magnitude nor direction depends on the location of the vector on the graph. Any vector drawn on the graph with that magnitude and direction will have the same length and the same orientation as the one already drawn there – we would say that the two vectors are equal.



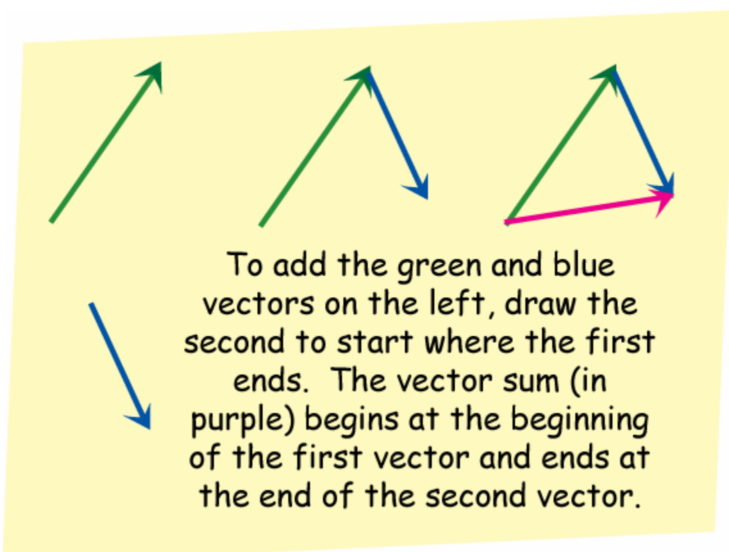
7.2 Vector Addition and Unit Vectors

Many ordinary mathematical operations that are normally applied to numbers can be applied to vectors as well. The simplest such operation is addition. You can't add a number to a vector, because any two things to be added together must be similar in form, and a number is just a number while a vector has both magnitude and direction.

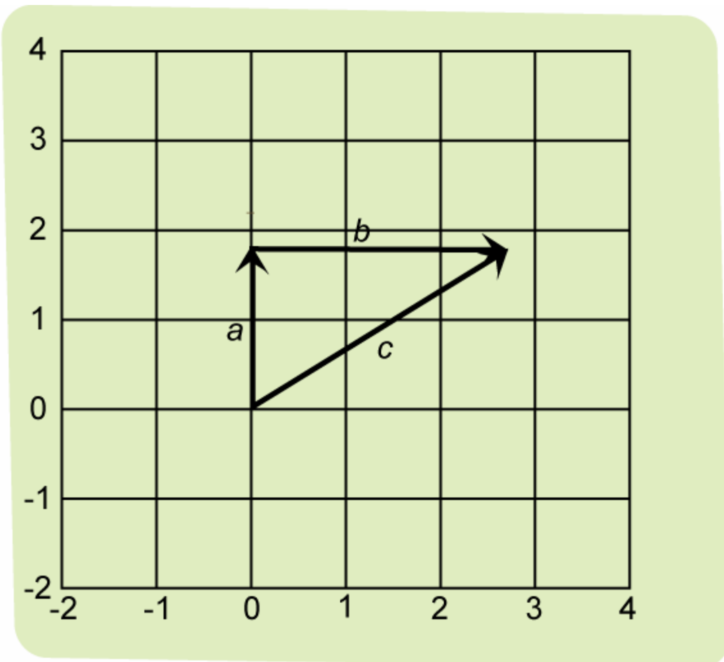
You may object that you add numbers to vectors all the time. For example, you might say that if you're driving at 55 miles per hour toward the north (a vector), and add 5 miles per hour (a scalar), you're then going 60 miles per hour toward the north. True, but what you added wasn't really a scalar, it was a vector, because that extra 5 miles per hour were directed toward the north. It had a direction as well as a magnitude.

Any two vectors can easily be added graphically. To do so, first draw the second vector so that it begins where the first vector ended. Then, a vector drawn from the beginning of the first vector to the end of the second vector is the sum of the two vectors.

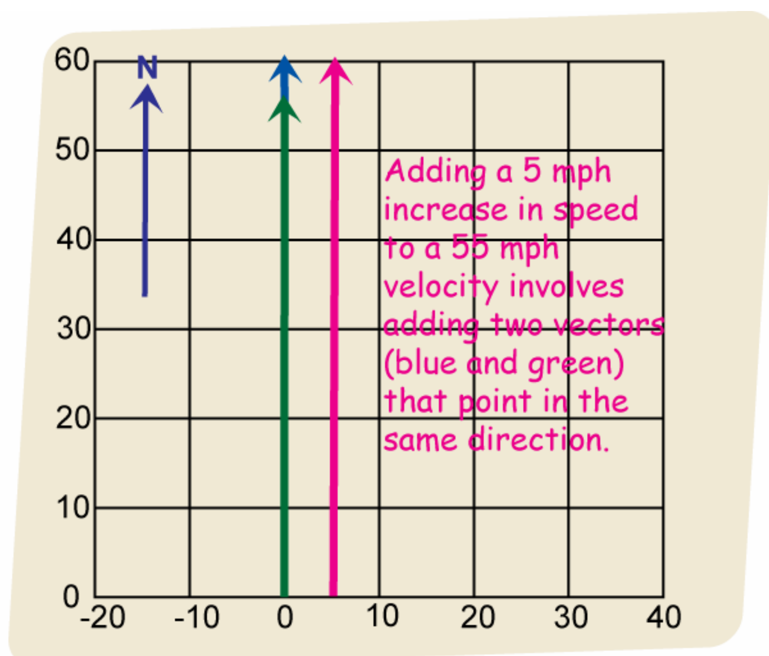
A visual example is shown here. To add the blue vector to the green vector, just move (or redraw) the blue vector so that it starts where the green vector left off. The sum of the two vectors is shown in purple: it starts at the beginning of the green vector and ends at the end of the blue vector.



Another example can be drawn from the previous section. Remember the components of the vector? We've redrawn it here. Suppose that the triangle side labeled *a* was really a vector that started at the origin and ended 1.8 grid boxes in the *y* direction. Also suppose that the triangle side labeled *b* was really a vector that started at the end of *a* and ended 2.8 grid points over in the x direction. Then the vector labeled $\boldsymbol{c}$ would be the sum of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.



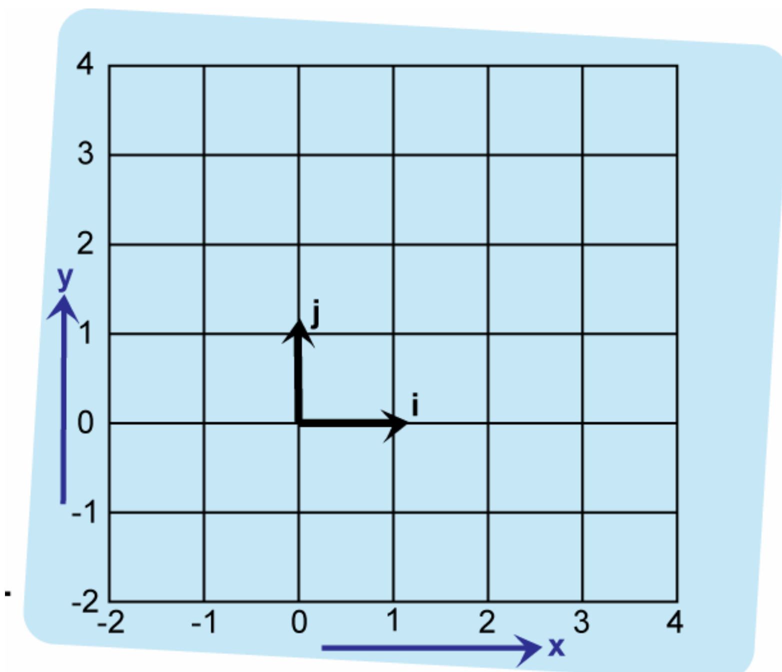
One last vector addition example is the car example from earlier in this section. It's kind of hard to draw a vector from the beginning of one to the end of the other when both vectors point in the same direction. So the diagram shows the sum vector off to one side. Also, so that the



vectors will fit on the page, the scale is different from the previous examples. A vector drawing is not much use without a scale: such a drawing shows direction but not magnitude.

7.3 Vector Multiplication and Components

Vectors oriented along the *x* and *y* axes come up so frequently that there's a simpler way to describe them using the concept of *unit vectors*. A unit vector is simply a vector with a magnitude of 1, in whatever units are currently popular. Three unit vectors have special names: unit vectors **i**, **j**, and **k** are the unit vectors oriented in the positive *x*, *y*, and *z* directions, respectively. The **i** and **j** unit vectors are shown here.



While you can't add a number to a vector, there's nothing wrong with multiplying a vector by a number. When you multiply a vector by some number, you simply multiply the magnitude of that vector by the number, without altering the vector's direction (unless the number is negative, in which case the new vector will point in the opposite direction).

What happens if you multiply a unit vector by a number? The general result is a vector with the magnitude of the number and the direction of the unit vector. (Unless the number is negative, in which case you get the direction opposite the unit vector.) For example, let's say you multiply the unit vector **j** by the number 0.9 m s^{-1}. Multiplication by a positive number doesn't affect the direction of the vector, so it's still pointing toward the north. Its magnitude is 1 x 0.9 m s^{-1}, or simply 0.9 m s^{-1}. This is vector ***a*** from earlier.

Vectors are often written in terms of components. Our vector ***a*** is 0.9 m s^{-1} **j**. This is precisely the multiplication in the previous paragraph. When 0.9 m s^{-1} is multiplied by **j**, you get vector ***a***. We would say that the **j** component of vector ***a*** is 0.9 m s^{-1} and that the **i** and **k** components are zero.

If an arbitrary vector does not happen to be parallel to one of the standard unit vectors, it will have more than one component. Vector ***c*** from above has components of 1.4 m s^{-1} **i** and 0.9 m s^{-1} **j**. Describing vector ***c*** as 1.4 m s^{-1} **i** and 0.9 m s^{-1} **j** (or, equivalently, 1.4 m s^{-1} **i** + 0.9 m s^{-1} **j**) conveys precisely the same amount of information as saying the vector ***c*** is 1.7 m s^{-1} from 238 degrees.

A component-based description of vectors is common in many circumstances and is especially useful for mathematical manipulations of vectors. Any vector in two- (or *n*-) dimensional space can be written as the sum of two (or *n*) component vectors oriented parallel to the axes. Indeed, it is so common when considering the wind vector that special variable names are applied to the magnitudes of the components of wind in the *x, y,* and *z* directions: *u, v,* and *w*. In the case of our favorite wind vector ***c***, we would say that u = 1.4 m s^{-1} and v = 0.9 m s^{-1}.

The simple non-graphical way of adding two vectors is to add the two vector components. Thus, if we had a wind vector of u = 2.5 m s^{-1} and v = 3.5 m s^{-1} and we wanted to add to it a wind vector of u = 1.5 m s^{-1} and v = -1 m s^{-1}, we could do it in our heads. The answer is a wind vector of u = 2.5 m s^{-1} + 1.5 m s^{-1} = 4 m s^{-1} and v = 3.5 m s^{-1} – 1 m s^{-1} = 2.5 m s^{-1}. Much easier than graphing the vectors, adding them that way, and then measuring the magnitude and direction of the sum.

Another very practical reason for expressing vectors using their components has to do with what happens when we remember that the universe we sense has three spatial dimensions, not just two. We have expressed the wind direction with respect to compass headings, but air is free to move up and down as well as sideways. Spatially speaking, the velocity of the air is a three-dimensional vector.

It's still possible to specify a direction in three dimensions, by giving the compass heading as well as the "elevation angle", the angle that the vector differs from horizontal. Astronomers specify angles this way all the time, as do radar scientists. But with component notation, it's really easy to include another dimension: just add the third component. Heck, if you want to, you can keep adding components until you get eight or ten dimensions. Don't laugh – physicists do this all the time.

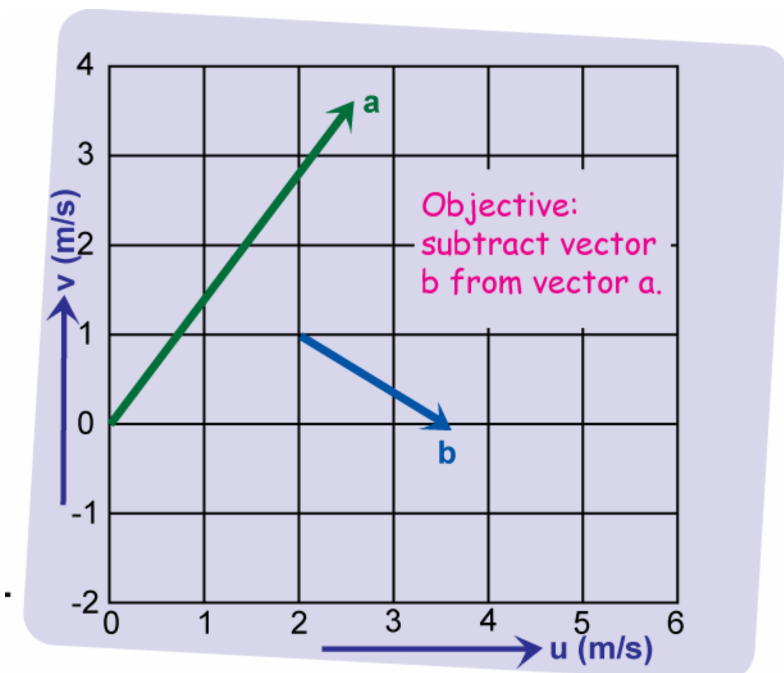
Aside from being convenient for mathematics and more than two dimensions, component notation for wind is of practical forecasting value.

Meteorologists care quite a bit about the sign and magnitude of w, because it is intimately related to the formation (or lack thereof) of clouds and precipitation. Specifically for forecasting, the horizontal components themselves have much less individual value. When working with wind in a forecasting environment, a hybrid strategy is most often used: specifying the horizontal part of the wind as a speed and direction and using w to specify the vertical wind.

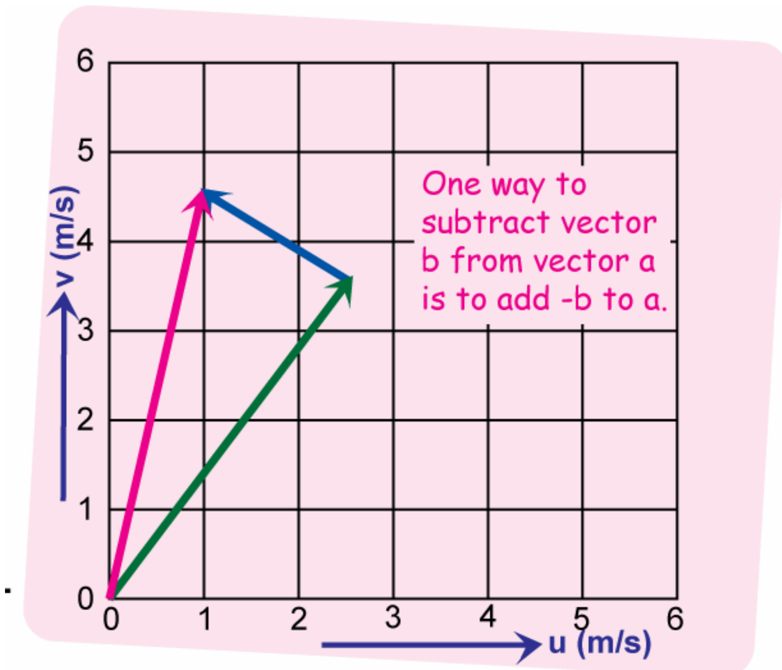
7.4 Vector Subtraction

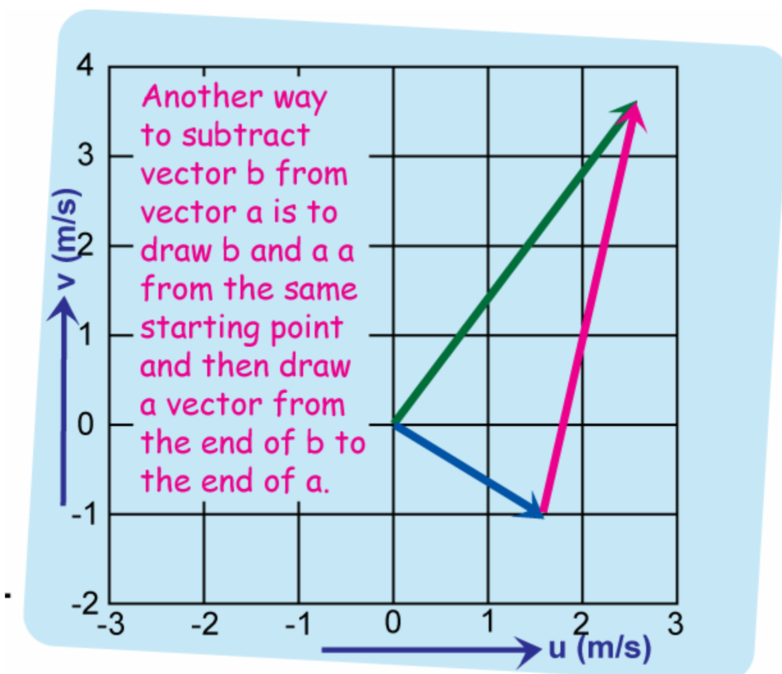
Vector subtraction always seems to give some students the heebie-jeebies. I'm not sure why. The act of subtracting two vectors using components is just as easy as adding two vectors. Graphically, there are two equally good ways to subtract vectors, while adding vectors only has one good way. Perhaps that's the problem: there are too many ways of getting the right answer, so students get confused.

To subtract vectors using components, just add subtract the components of the second vector from the components of the first vector. So, keeping with our previous example, if we want $\boldsymbol{a} - \boldsymbol{b}$, we can compute a u of 2.5 m s^{-1} – 1.5 m s^{-1} = 1 m s^{-1} and a v of 3.5 m s^{-1} –(-1 m s^{-1}) = 4.5 m s^{-1}.



One graphical way to subtract vectors is to recognize that $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$. So just draw $\boldsymbol{b}$ backwards and add it to $\boldsymbol{a}$. The other graphical way is just as simple: just draw both vectors starting at the origin, then draw a third vector starting at the endpoint of ***b*** and ending at the endpoint of ***a***. This third vector is ***a*** – ***b***. Notice from the pictures that both techniques yield the same answer.





An important application of vector subtraction is wind shear. Frequently it is important to know the difference in wind between one level and another. Determining the difference between the vector wind at one level and that at another requires vector subtraction.

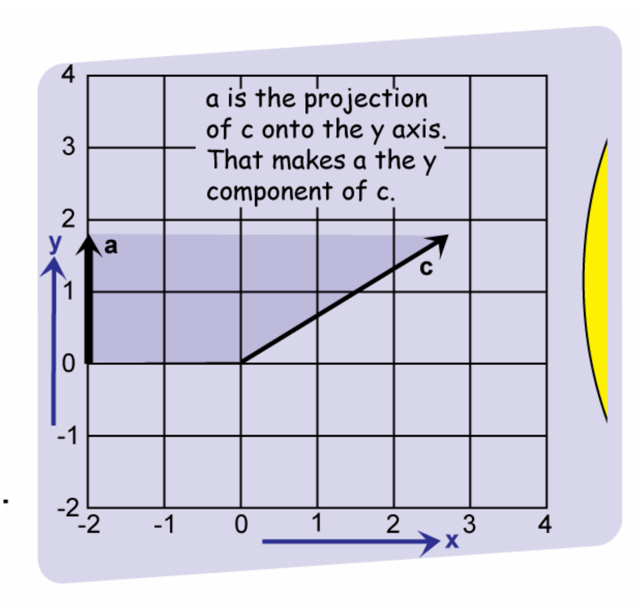
7.5 The Dot Product

There are two other vector operations that are common in meteorology, as well as anywhere else that uses vectors: the dot and cross product. The use of the term "product" seems to imply multiplication, and indeed the computation of dot and cross products using components involves multiplication. However, I personally don't think that multiplication is the right way to think about these operations in a meteorological setting.

Take the dot product. Let's write the *u* and *v* wind components of a vector ***a*** as u_a and v_a. Expressed as components, the dot product of horizontal wind vectors ***a*** and ***b*** is $u_a u_b + v_a v_b$. If you have three or more components to your vectors, just keep adding the products of the individual components until you run out of components.

Expressed as magnitudes and directions, the dot product is $|\boldsymbol{a}|\ |\boldsymbol{b}|\cos\theta$, where $|\boldsymbol{a}|$ and $|\boldsymbol{b}|$ are the magnitudes of the two vectors and θ is the angle between them.

Note that the dot product is a single number (a scalar) rather than two components or a magnitude and direction (a vector), so the dot product is often referred to as the scalar product.



So far, what we've said about the dot product is merely annoying fodder for memorization. The dot product takes on physical significance, though, if we keep in mind that the dot product of some vector with a unit vector is the projection of that vector onto the axis represented by the unit vector. Imagine that there was a light off way in the distance to the right of the first figure in section 7.1. The line labeled *a* would be the shadow cast by the vector ***c*** onto the *y* axis. We say that *a* is the *projection* of ***c*** onto *y*.

The length of this line is computed as (and is equivalent to) the dot product $\boldsymbol{c} \bullet \boldsymbol{j}$. In components, we have (1.4 m s^{-1} * 0) + (0.9 m s^{-1} * 1) = 0.9 m s^{-1}. This is exactly equal to the *v* component of ***c***. Similarly, if we compute the dot product using magnitudes and directions, we get 1.6 m s^{-1} * 1 * cos 58 = 0.9 m s^{-1}.

In summary, the dot product is a measure of the magnitudes of two vectors and the smallness of the angle between them. The smallness is measured by the cosine of the angle; if the two vectors are at right angles, the cosine is 0 and so is the dot product. The dot product can be thought of as the projection of one vector onto another, multiplied by the magnitude of that second vector.

7.6 Advection as a Dot Product

What does it take to get a large dot product? For starters, it helps if the two vectors being "multiplied" together are large rather than small. Also, the angle between the two vectors should be as small as possible; if the two vectors are parallel, the dot product is largest.

Does that sound familiar? Remember advection: advection is strongest if the wind is strong and the gradient of the thing being advected is also strong. Furthermore, the wind should be close to perpendicular to the isopleths of the advectee, that is, close to parallel to the gradient vector of the advectee. Those are the same principles as for a dot product, so it should not be surprising that advection actually *is* a dot product.

The advection of some scalar quantity *T* (such as temperature) is written as:

$$-\mathbf{v} \bullet \nabla_h T$$

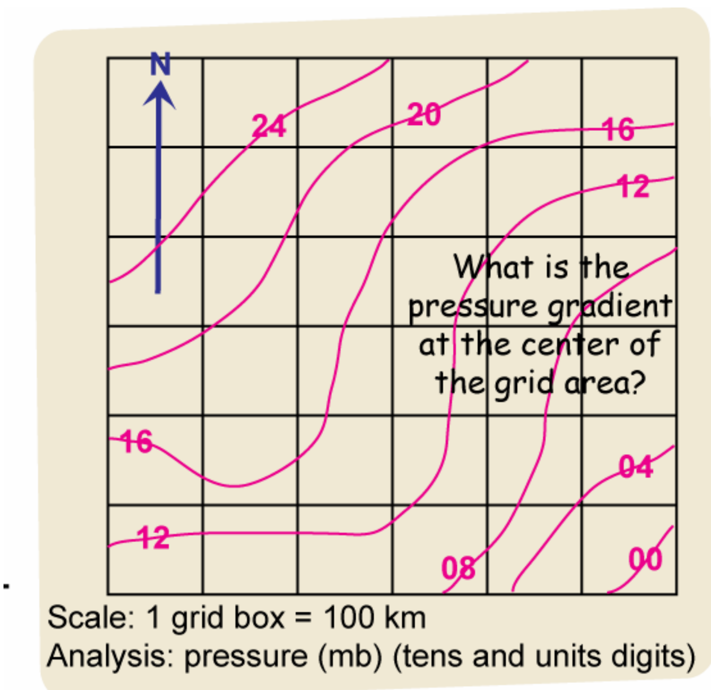
This is a new equation, so first let's check to see if it makes sense. The gradient vector $\nabla_h T$ points across isopleths (isotherms) toward higher values (of temperature). If the wind is blowing in the same direction, the dot product of the two vectors will be largest, and meanwhile the wind will cause the temperatures to drop. This is cold advection, and the minus sign in the definition of advection indeed produces a negative value in this example.

Let's explore the equation a bit more. While gradient notation is useful for writing things concisely, it takes a while to get used to the meaning, so let's figure out how the definition of advection works when the vectors are written in component notation.

First, the gradient $\nabla_h T$. The components of this vector are defined as

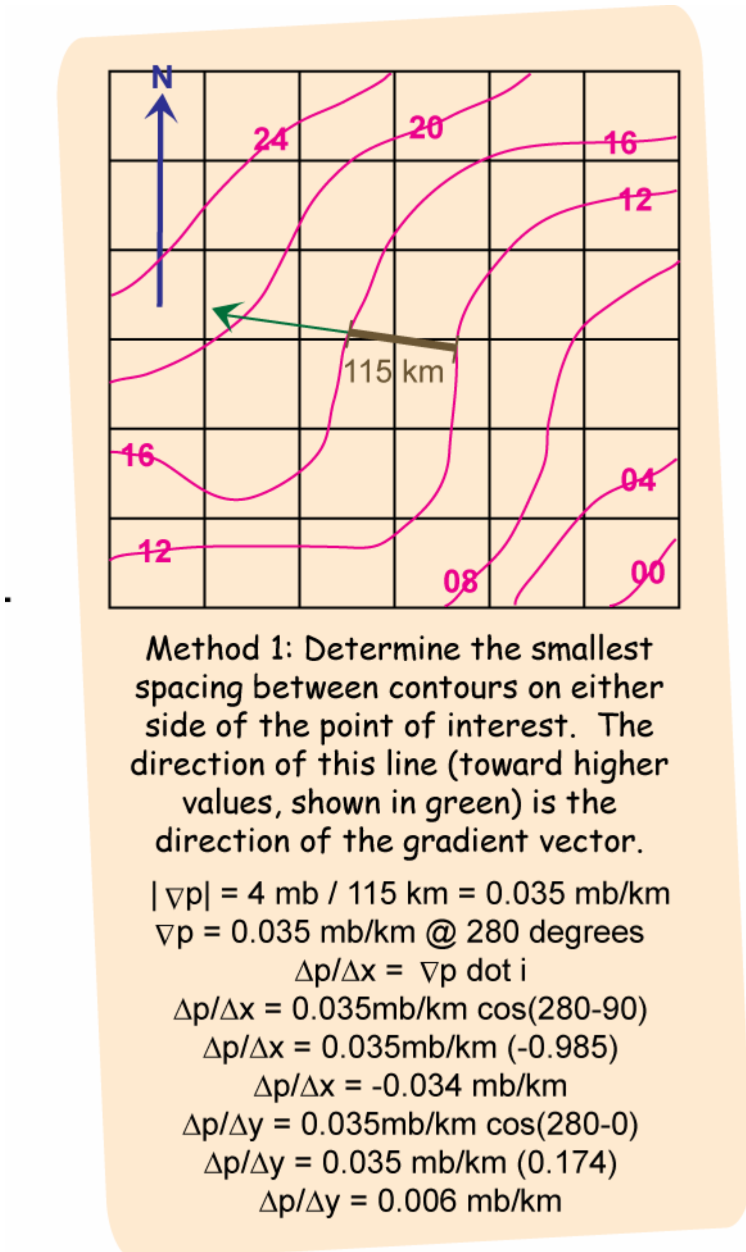
$$\nabla_h T \equiv \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}$$

Each component of the gradient vector is equal to the rate at which *T* changes in the corresponding direction. For example, the *x* component is equal to the rate at which *T* changes in the *x* direction.



There are several ways of estimating gradients from a map with isopleths. Here these methods are illustrated using a sample pressure analysis. By convention, the isobars are labeled with only the tens and units digits; the leading "10" (in this case) is dropped.

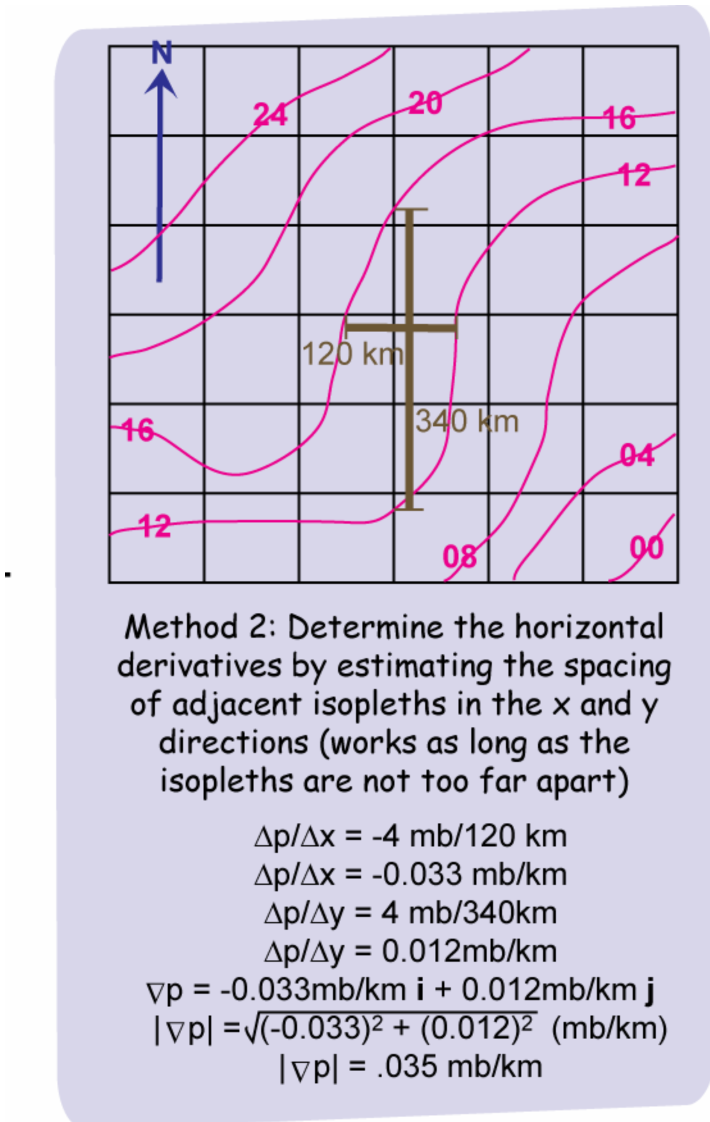
All methods require knowing the horizontal scale of the map. Beyond this, you can either estimate the magnitude and direction of the gradient directly (and then determine the components of the gradient vector as the projections on x and y) or estimate the two components of the gradient vector and then compute the magnitude of the gradient using the Pythagorean Theorem. You could also compute the direction of the gradient using trig, but an exact direction is rarely needed in practice.



Okay, say we've computed the horizontal gradient of temperature using one of the three methods. For computing advection, we also need the horizontal wind **v**. We already know the components of **v**: $u\mathbf{i} + v\mathbf{j}$. To get the advection, multiply the corresponding components together and throw in a minus sign:

$$-\mathbf{v} \bullet \nabla_h T = -u\frac{\partial T}{\partial x} - v\frac{\partial T}{\partial y}$$

We've seen something like that equation before. Suppose we choose to orient the coordinate system so that the *x* axis is parallel to the

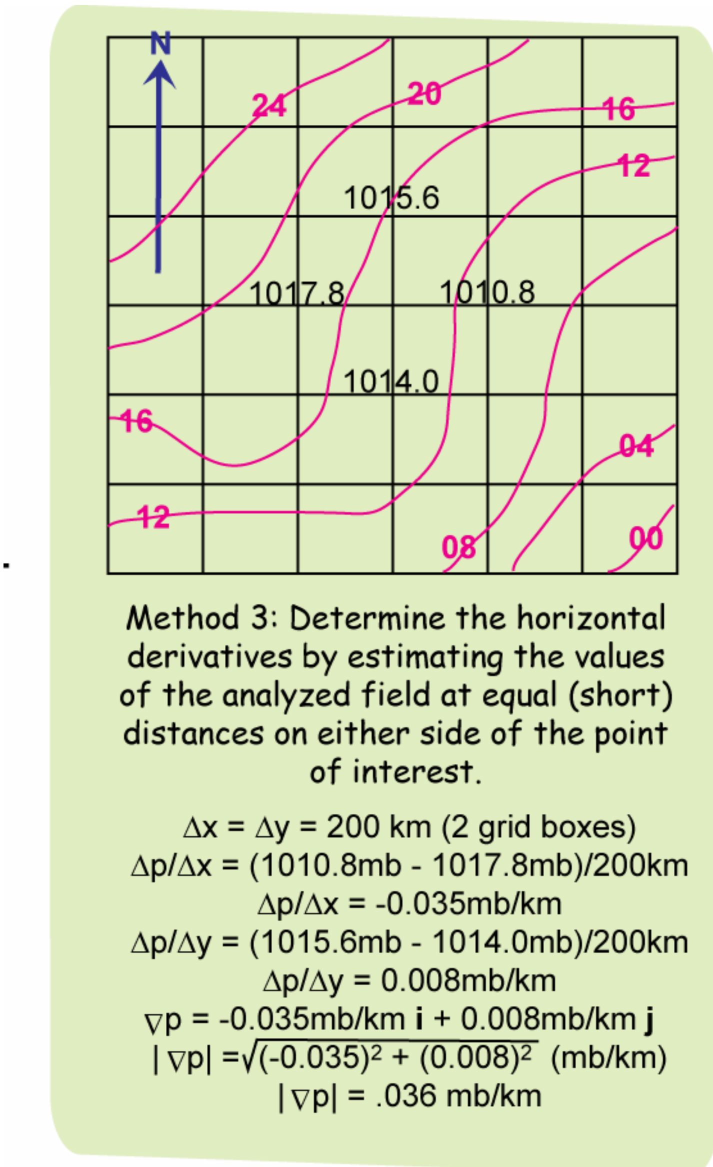


wind. Then *v* is zero and the second term on the right-hand side vanishes. What's left is the same equation that was introduced in Chapter 3. Back then, we stated that the equation only worked if *x* is chosen to be parallel to the wind. That's true, because only then is *v* zero.

One could compute the advection using that equation, but when you're eyeballing advection on a map it's much easier to use the alternative way of writing the dot product: $|\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$. For advection, this is

$$-\mathbf{v}\bullet\nabla_h T = -|\mathbf{v}||\nabla_h T|\cos(\theta)$$

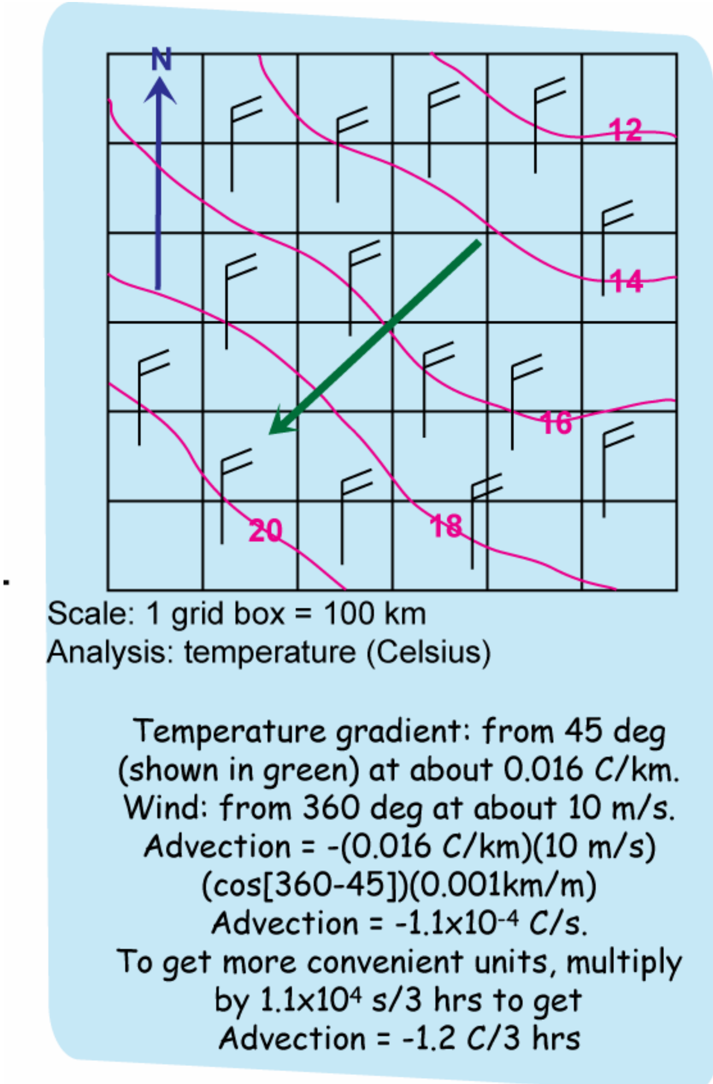
Just take the wind speed, multiply by the temperature gradient, and finally multiply by the cosine of the difference in angle between the wind vector and temperature gradient vector.



An example is shown in the figure on the next page. It doesn't hurt to have a few key values of cosine memorized: cos(0) = 1, cos(60) = 0.5, cos(90) = 0, cos(120) = -0.5, and cos(180) = -1.

7.7 The Cross Product

The *cross product* is, like the dot product, a measure of the magnitudes of two vectors and the largeness of the angle between them. The largeness is measured by the sine of the angle; if the two vectors are parallel, the sine is 0 and so is the cross product. So the cross product is sort of the opposite of the dot product.

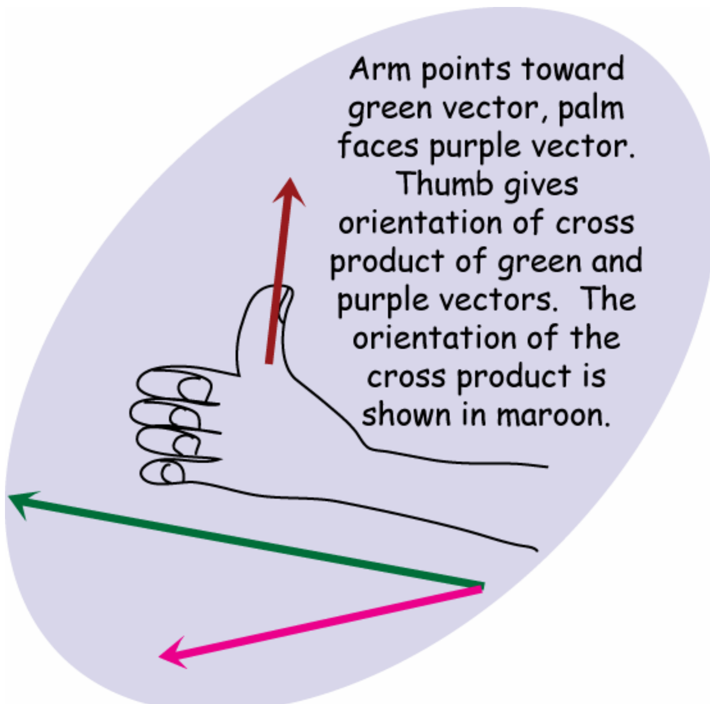


One thing that makes the cross product a bit more complicated than the dot product is that the answer is a vector rather than a scalar. The mathematical techniques for computing the complete cross product are sufficiently involved that we won't go into them here. Fortunately for us, the magnitude of the cross product vector is fairly simple. As you might imagine from the above description, the magnitude is $|\boldsymbol{a}|\ |\boldsymbol{b}| \sin\theta$.

For future reference, we'll note one property of this magnitude computation here. If one of the vectors is a unit vector oriented perpendicular to the other vector, the magnitude of the cross product is simply the magnitude of the other vector.

The direction of the cross product vector is a bit stranger. The cross product vector is perpendicular to the plane on which the two vectors being "crossed" lie. But you don't really need to know that now. What you do need to know is the rule for graphically determining the orientation of the cross product of two vectors.

At most schools, they use something called the “right-hand rule”. Here in Aggieland, we use the “Gig ‘em” rule. Begin by making the Gig ‘em signal with your right hand. Next, point your fist in the same direction as the first vector. Then, without changing the direction of your forearm, rotate your thumb so that the palm of your hand is facing the direction toward which the second vector points. The direction of the cross product is now given as the orientation of your right thumb.



I bet you always thought that the “Gig ‘em” signal was just a pleasant but meaningless way of expressing Aggie spirit. Now you know the truth. It’s really our way of saying to the rest of the world, “Hah! *We* know how to determine a cross product!” Take pride in your mathematics!

Unlike the dot product, you may be able to survive a meteorology curriculum without fully understanding the cross product.

7.8 Divergence and Curl

There are a couple of calculus operations that share the notations of dot and cross products. For reference, we list them here, but we won’t make use of them for a few more chapters.

Although these operations can be applied to any vector, most common meteorological applications of them involve the horizontal wind. So, we'll always work with wind here.

The *horizontal divergence* is defined as

$$\nabla_h \bullet \mathbf{v} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

A related quantity, the three-dimensional divergence, is

$$\nabla \bullet \mathbf{v} \equiv \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The vertical component of the curl of the horizontal wind is called the *relative vorticity*. It is defined as

$$\mathbf{k} \bullet (\nabla \times \mathbf{v}) \equiv \varsigma = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Questions

1. Sketch arrows with the following direction and magnitude on a piece of paper. Use a consistent horizontal scale for each. The directions are provided using the wind convention: the direction the vectors are pointing *from*. a) 270 deg, 5 m/s b) 330 deg, 8 m/s c) 196 deg, 1 m/s d) 48 deg, 12 m/s e) 92 deg, 2 m/s

2. Compute the components of the vectors in Question 1.

3. Compute vectors a+b, b+c, c+d, d+e, and e+a from Question 1.

4. Repeat question 3 using purely graphical techniques and determine the accuracy of your answers.

5. Compute vectors a-b, b-c, c-d, d-e, and e-a from Question 1.

6. Repeat question 5 using purely graphical techniques and determine the accuracy of your answers.

7. Compute the dot products of a&b, b&c, c&d, d&e, and e&a from Question 1.

8. If the horizontal temperature gradient is oriented toward the SE at 2 C/100 km, compute the temperature advection with winds a through e from question 1.

9. List all the possible directions that the cross product of two horizontal vectors can point.